Spectral statistics of the *k*-body random-interaction model

Mark Srednicki
Department of Physics, University of California, Santa Barbara, California 93106
(Received 9 July 2002; published 30 October 2002)

We reconsider the question of the spectral statistics of the *k*-body random-interaction model, investigated recently by Benet, Rupp, and Weidenmüller, who concluded that the spectral statistics are Poissonian. The binary-correlation method that these authors used involves formal manipulations of divergent series. We argue that Borel summation does not suffice to define these divergent series without further (arbitrary) regularization, and that this constitutes a significant gap in the demonstration of Poissonian statistics. Our conclusion is that the spectral statistics of the *k*-body random-interaction model remains an open question.

DOI: 10.1103/PhysRevE.66.046138 PACS number(s): 02.50.Ey, 05.45.−a, 21.10.−k, 24.60.Lz

I. INTRODUCTION

In recent work, Benet, Rupp, and Weidenmüller (hereafter BRW) [1,2] have considered the spectral statistics of *k*-body random interaction models [3–5]. These are models of fermions (usually; boson versions can also be studied [6]) in which there are ℓ single-particle states occupied by a total of *m* particles. The models are specified by Hamiltonians of the form

$$H_k=\sum_{\substack{1\leqslant j_1<j_2<\cdots<j_k\leqslant\ell\\ 1\leqslant i_1<i_2<\ldots<i_k\leqslant\ell}} V_{j_1,\ldots,j_k,i_1,\ldots,i_k}a^\dagger_{j_1}\cdots a^\dagger_{j_k}a_{i_k}\cdots a_{i_1}, \tag{1}$$

where the $a^\dagger$'s and a's are creation and annihilation operators, and the V's are (for the case of the unitary ensemble) complex numbers that obey

$$V_{j_1,\ldots,j_k,i_1,\ldots,i_k}=V^*_{i_1,\ldots,i_k,j_1,\ldots,j_k}. \tag{2}$$

Each independent component is a Gaussian random variable with mean zero and variance v_{0k}^2,

$$\overline{V_{j_1,\ldots,j_k,i_1,\ldots,i_k}V_{j'_1,\ldots,j'_k,i'_1,\ldots,i'_k}} =v_{0k}^2\delta_{j_1i'_1}\cdots\delta_{j_ki'_k}\delta_{i_1j'_1}\cdots\delta_{i_kj'_k}. \tag{3}$$

The overbar denotes averaging over the ensemble, and v_{0k} is a normalization constant that sets the energy scale. In this paper, for pedagogical simplicity, we will restrict our attention to the unitary ensemble.

These models can be viewed as caricatures of complex systems of interacting particles, such as nuclei, multielectron atoms, or quantum dots. The most physically interesting case is $k=2$: H_2 can be thought of as representing two-body interactions among a set of *m* particles occupying ℓ degenerate single-particle states.

In their work, BRW analyzed the spectral statistics of these models, and found that they are Poissonian for $\ell\gg m\gg k$. This is something of a surprise; numerical simulations of the $k=2$ model have generally found Wigner-Dyson spectral statistics (see, e.g., [7,8]). Even more surprising, the proof of this offered by BRW extends trivially to the case of nondegenerate single-particle levels specified by $H=H_1+H_2$, for *arbitrary* values of v_{02}/v_{01} [9].

This is counterintuitive on at least two levels. First, mathematically, one would naturally expect that determining the spectral statistics of H_1+H_2 would be a much harder problem than it is for either H_1 or H_2 alone. Second, physically, we expect this model to capture the essential physics of a chaotic many-body system (with two-body interactions), just as a single random matrix is often a good model of a chaotic few-body system. According to the Bohigas-Giannoni-Schmit conjecture [10] (see also [11]), a classically chaotic system should exhibit Wigner-Dyson statistics at sufficiently high energy (irrespective of the number of degrees of freedom). If the BRW result is correct, we must then conclude that either the BGS conjecture is false for many-body systems, or that the two-body random-interaction model does not correctly capture this aspect of the physics of these systems.

In this paper we will show that there is in fact a gap in the BRW proof, and that this gap is not easily bridged. Thus, we will argue, Wigner-Dyson statistics for these models is still an open possibility.

In Sec. II we review and expand the analysis of BRW [1,2], and explain how their proof of Poissonian spectral statistics might fail to hold. In Sec. III we specialize in the case $k=1$ (where the detailed analysis simplifies) and show that, within the context of the binary correlation method used by BRW, the proof does break down in the manner suggested in Sec. II. However, the binary correlation method is not exact, and the appropriate conclusion is that one must go beyond this approximation to obtain a reliable evaluation of the spectral statistics of these models. We elaborate on this further in Sec. IV.

II. THE BINARY CORRELATION METHOD

We begin with the resolvent

$$G(z)=\mathrm{tr}\frac{1}{z-H}, \tag{4}$$

where tr denotes the normalized trace ($\mathrm{tr}1=1$) over states of *m* particles. BRW compute the connected correlation function

$$R(z_1,z_2)=\overline{G(z_1)G(z_2)}-\overline{G(z_1)}\;\overline{G(z_2)}. \tag{5}$$

Then, since the density of states $\rho(E)$ is given by

$$\rho(E)=\frac{1}{2\pi i}[G(E-i\varepsilon)-G(E+i\varepsilon)], \tag{6}$$

where E is real and ε is a positive infinitesimal, we have

$$\begin{aligned}\overline{\rho(E_1)\rho(E_2)}&-\overline{\rho(E_1)}\;\overline{\rho(E_2)}\\ &=\frac{1}{(2\pi)^2}[R(z_1^+,z_2^-)+R(z_1^-,z_2^+)-R(z_1^+,z_2^+)\\ &\quad -R(z_1^-,z_2^-)],\end{aligned} \tag{7}$$

where $z_i^{\pm}=E_i\pm i\varepsilon$. Thus, we can extract the connected density-density correlation (which provides information on the spectral statistics) from $R(z_1,z_2)$, *provided* we can evaluate $R(z_1,z_2)$ with both z_1 and z_2 on either side of the real axis.

To compute $R(z_1,z_2)$, BRW expand in powers of H,

$$\overline{G(z_1)G(z_2)}=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{z_1^{r+1}z_2^{s+1}}\overline{\mathrm{tr}(H^r)\mathrm{tr}(H^s)}. \tag{8}$$

The ensemble average vanishes unless $r+s$ is even, and, in this case, Wick's theorem can be used to express it as a sum over $(r+s-1)!!$ pairwise contractions of V's. Each contraction that occurs inside one of the two traces can be shown [2] to yield a factor of $v_{0k}^2\Lambda_k$, where Λ_k is a calculable number (given below), provided the total number of these contractions is much less than ℓ. This is the essence of the binary correlation method, introduced by Mon and French [5] (see also [12]). It is convenient to choose v_{0k} so that $v_{0k}^2\Lambda_k=1$; then we have

$$v_{0k}^{-2}=\Lambda_k=\binom{m}{k}\binom{\ell-m+k}{k}. \tag{9}$$

It remains to count the total number of contractions within each trace, and to evaluate the contractions across the two traces. Performing the first task yields [2]

$$R(z_1,z_2)=\sum_{n=1}^{\infty}g_n(z_1)g_n(z_2)T_n, \tag{10}$$

where

$$g_n(z)=\sum_{p=0}^{\infty}\frac{(2p-1)!!}{z^{2p+n+1}}\binom{n+2p}{n} \tag{11}$$

and

$$T_n=\overline{\overline{\mathrm{tr}(H^n)\mathrm{tr}(H^n)}}; \tag{12}$$

the double overbar means that all contractions of V's are to involve one V from each of the two traces. Equations (10)–(12) are equivalent to Eq. (68) of [2].

BRW show that $T_n\sim\ell^{-kn}$ for $\ell\gg m\gg k$, and so conclude that $R(z_1,z_2)$ vanishes in the $\ell\to\infty$ limit; this is indicative of Poissonian spectral statistics. However, this conclusion is suspect if we do not first understand the convergence properties of the series in Eq. (10). To see why this is necessary, consider the mathematical example

$$S=\sum_{n=1}^{\infty}\frac{n!}{\ell^n}, \tag{13}$$

where ℓ is a positive integer. Clearly, in the limit $\ell\to\infty$, each term of this series vanishes, and so it is tempting to conclude that $S=0$. However, we can evaluate S by Borel summation. We use

$$n!=\int_0^{\infty}dte^{-t}t^n \tag{14}$$

in Eq. (13) and do the sum to get

$$S=\int_0^{\infty}dte^{-t}\frac{t}{\ell-t}. \tag{15}$$

This integral is not defined for positive real ℓ. We could attempt to define it by analytic continuation, but there would still be an ambiguity, corresponding to whether the positive real axis is approached from above or below. More importantly, in our case ℓ is a positive integer that counts the number of single particle states; therefore it does not seem to make sense to analytically continue to complex ℓ.[1] Our conclusion in this case would be that S is simply not defined by the Borel procedure. We wish to examine whether the same problem arises for the series in Eq. (10).

The series for $g_n(z)$ in Eq. (11), on the other hand, can be defined for any z with $\mathrm{Im}z\neq 0$ by Borel summation followed by analytic continuation. To demonstrate this, we begin with the combinatoric identities

$$\binom{n+2p}{n}=\frac{(n+2p)!}{n!(2p)!}\quad\text{and}\quad(2p-1)!!=\frac{(2p)!}{2^p p!}, \tag{16}$$

and so

$$g_n(z)=\frac{1}{n!z^{n+1}}\sum_{p=0}^{\infty}\left(\frac{1}{2z^2}\right)^p\frac{(n+2p)!}{p!}. \tag{17}$$

We now use Eq. (14) with $n\to n+2p$ to get

[1] One could raise the same objection to dimensional regularization in quantum field theory, but there the results can be verified by a variety of different and more physically motivated schemes.

$$g_n(z)=\frac{1}{n!z^{n+1}}\sum_{p=0}^{\infty}\int_0^{\infty}dte^{-t}\left(\frac{t^2}{2z^2}\right)^p\frac{t^n}{p!}$$

$$=\frac{1}{n!z^{n+1}}\int_0^{\infty}dte^{-t}t^ne^{t^2/2z^2}. \tag{18}$$

The integral converges provided $\mathrm{Re}z^{-2}<0$. This condition is satisfied for $z=\pm i|z|e^{i\phi}$ with $-\frac{1}{4}\pi<\phi<+\frac{1}{4}\pi$. We now rotate the t contour in the complex t plane so that it runs along a straight line from zero to complex infinity at an angle of ϕ relative to the positive real axis. Then we set $t=\tau e^{i\phi}$, where τ is real and runs from zero to infinity. We now have

$$g_n(z)=\frac{e^{i(n+1)\phi}}{n!(\pm i)^{n+1}|z|^{n+1}e^{i(n+1)\phi}}$$

$$\times\int_0^{\infty}d\tau e^{-\tau e^{i\phi}}\tau^ne^{-\tau^2/2|z|^2}, \tag{19}$$

where the phase in the numerator of the prefactor comes from the change of variable $t\rightarrow\tau e^{i\phi}$, and the phase in the denominator comes from $z=\pm i|z|e^{i\phi}$. We now change the integration variable to $u=\tau/|z|$ to get

$$g_n(z)=\frac{(\mp i)^{n+1}}{n!}\int_0^{\infty}due^{-u|z|e^{i\phi}}u^ne^{-u^2/2}$$

$$=\frac{(\mp i)^{n+1}}{n!}\int_0^{\infty}due^{\pm izu}u^ne^{-u^2/2}. \tag{20}$$

This integral converges for all z, and so constitutes an analytic continuation of Eq. (18); the $\pm$ symbol should be interpreted as the sign of Imz. Thus $g_n(z)$ is discontinuous across the real axis, and if we take $z=z^{\pm}=E\pm i\varepsilon$, we get

$$g_n(z^{\pm})=\frac{(\mp i)^{n+1}}{n!}\int_0^{\infty}due^{\pm iEu}u^ne^{-u^2/2}. \tag{21}$$

We also note that, from our Eq. (11) and Eq. (65) of [2],

$$\overline{G(z)}=g_0(z). \tag{22}$$

Then, from Eq. (6) and Eq. (21), we have

$$\overline{\rho(E)}=\frac{1}{2\pi i}[g_0(z^-)-g_0(z^+)]$$

$$=\frac{1}{2\pi}\int_0^{\infty}du(e^{-iEu}+e^{+iEu})e^{-u^2/2}$$

$$=\frac{1}{\sqrt{2\pi}}\exp(-E^2/2). \tag{23}$$

This is the classic result of Mon and French [5]: the ensemble-averaged density of states of the k-body random-interaction model is Gaussian.

Returning to $R(z_1,z_2)$, we use Eqs. (10) and (20) to get

$$R(z_1^+,z_2^{\pm})=\mp\int_0^{\infty}dudve^{i(E_1u\pm E_2v)}e^{-(u^2+v^2)/2}F(\mp uv), \tag{24}$$

where we have defined

$$F(y)\equiv\sum_{n=1}^{\infty}\frac{y^n}{(n!)^2}T_n. \tag{25}$$

We see that, in order for both $R(z_1^+,z_2^+)$ and $R(z_1^+,z_2^-)$ to be well defined, $F(y)$ must be free of singularities on the real axis (positive or negative). Only then can we perform the integrals over u and v in Eq. (24) for both $R(z_1^+,z_2^+)$ and $R(z_1^+,z_2^-)$ without further (arbitrary) regularization.

To see whether or not this obstacle arises, we must evaluate T_n. Here we have an immediate difficulty. We have already invoked the binary correlation approximation in Eq. (11); the terms in this series receive corrections when the summation index p becomes comparable to ℓ, and there is no straightforward way to calculate these corrections exactly. Similarly, in Sec. III we will evaluate T_n (for $k=1$), but our method will require $n\ll\ell$. It is therefore unsuitable for reliably determining the asymptotic behavior of the series in Eq. (25). Still, it is worthwhile to see whether or not the problem of a singularity on the real axis arises within this approximation. We therefore turn to the calculation of T_n for $k=1$.

III. ANALYSIS FOR $k=1$

We specialize in the case $k=1$ (and drop the corresponding "1" subscripts):

$$H=\sum_{j,i}V_{ji}a_j^{\dagger}a_i, \tag{26}$$

with

$$\overline{V_{ji}V_{j'i'}}=v_0^2\delta_{ji'}\delta_{ij'}. \tag{27}$$

Of course, we already know the answer for this case: the spectral statistics are Poissonian, because the spectrum simply consists of the linear sum of the m single-particle energies that are obtained by diagonalizing the $\ell\times\ell$ Hermitian matrix V. These single-particle energies obey Wigner-Dyson statistics, but their sums (for $m\gg1$) obey Poisson statistics. However, the binary correlation method can still be used, and it is important to see whether or not it gives the correct answer (or any answer at all).

We first introduce some shorthand notation. Let $I=\{i_1,\ldots,i_n\}$ and $J=\{j_1,\ldots,j_n\}$; let

$$\delta_{I,J}=\delta_{i_1j_1}\cdots\delta_{i_nj_n}. \tag{28}$$

Let P, Q, and R denote permutations of the n indices of I. Then, from Eq. (3), we can write

$$\overline{\overline{(V_{j_1 i_1}\cdots V_{j_n i_n})(V_{j'_1 i'_1}\cdots V_{j'_n i'_n})}} = v_0^{2n}\sum_R \delta_{I,RJ'}\delta_{J,RI'}. \tag{29}$$

Another useful bit of shorthand is

$$A_J^\dagger A_I = a_{j_1}^\dagger a_{i_1}\cdots a_{j_n}^\dagger a_{i_n}. \tag{30}$$

Thus Eq. (12) becomes

$$T_n = v_0^{2n}\sum_R \sum_{IJI'J'} \delta_{I,RJ'}\delta_{J,RI'}\mathrm{tr}(A_J^\dagger A_I)\mathrm{tr}(A_{J'}^\dagger A_{I'}). \tag{31}$$

The trace is a $U(\ell)$ invariant operation, and so we must have

$$\mathrm{tr}(A_J^\dagger A_I) = \sum_P C_P \delta_{J,PI} \tag{32}$$

for some set of coefficients C_P. Substituting this expansion into Eq. (31), we get

$$\begin{aligned} T_n &= v_0^{2n}\sum_{P,Q,R} C_P C_Q \sum_{IJI'J'} \delta_{I,RJ'}\delta_{J,RI'}\delta_{J,PI}\delta_{J',QI'} \\ &= v_0^{2n}\sum_{P,Q,R} C_P C_Q \sum_I \delta_{I,RQ\bar{R}PI}, \end{aligned} \tag{33}$$

where $\bar{R}$ is the inverse of R.

The expression $\Sigma_I \delta_{I,PI}$ can be evaluated in terms of the cycles of P. Consider, for example, the permutation 2431 of 1234; it contains one cycle of length one (since 3 remains in its original position), and one cycle of length three (since 1 is replaced by 2, 2 is replaced by 4, and 4 is replaced by 1). Let $\gamma_c(P)$ denote the number of cycles of length c in permutation P; for our example, $\gamma_1=\gamma_3=1$ and $\gamma_2=\gamma_4=0$. Since each element of any permutation is in exactly one cycle, we have $\Sigma_{c=1}^n \gamma_c(P)c = n$. In $\Sigma_I \delta_{I,PI}$, each cycle of P ultimately results in a factor of $\Sigma_i \delta_{ii} = \ell$. Thus we have

$$\sum_I \delta_{I,PI} = \ell^{\Sigma_c \gamma_c(P)}, \tag{34}$$

and hence

$$T_n = v_0^{2n}\sum_{P,Q,R} C_P C_Q \ell^{\Sigma_c \gamma_c(RQ\bar{R}P)}. \tag{35}$$

So far we have made no approximations in our evaluation of T_n. We now notice that, for $\ell \gg 1$, the dominant term on the right-hand side of Eq. (35) is the one with the largest value of $\Sigma_c \gamma_c(RQ\bar{R}P)$. This occurs when $\gamma_1 = n$ and $\gamma_c = 0$ for $2 \leqslant c \leqslant n$, which in turn implies $RQ\bar{R}P = I$, or $Q = \bar{R}P\bar{R}$. Thus we have

$$T_n \cong v_0^{2n}\ell^n \sum_{P,R} C_P C_{\bar{R}P\bar{R}}, \tag{36}$$

where $\cong$ denotes equality in the limit of large ℓ.

Next we must evaluate C_P. Starting with Eq. (32), we set $J = QI$ and sum over I to get

$$\begin{aligned} \sum_I \mathrm{tr}(A_{QI}^\dagger A_I) &= \sum_P C_P \sum_I \delta_{QI,PI} \\ &= \sum_P C_P \ell^{\Sigma_c \gamma_c(\bar{Q}P)} \\ &\cong C_Q \ell^n. \end{aligned} \tag{37}$$

To get the last line, we used the same large-ℓ argument that gave us Eq. (36).

Now we need to evaluate $\Sigma_I \mathrm{tr}(A_{PI}^\dagger A_I)$. To do so, imagine rearranging the $a^\dagger$'s so that their indices are in the same order as the a's, and interleaved among them in the standard pattern of Eq. (30). Since $a_i^\dagger a_j^\dagger = -a_j^\dagger a_i^\dagger$, we see that, before accounting for the presence of the a's, rearranging the $a^\dagger$'s in this way will yield a factor of $(-1)^P$. Passing $a^\dagger$'s through a's yields Kronecker deltas, since $a_i a_j^\dagger = -(a_j^\dagger a_i - \delta_{ij})$, but no possibility of an extra overall minus sign, since the total number of exchanges of $a^\dagger$'s with a's is always even. Let us temporarily ignore the Kronecker deltas. Then, once all the $a^\dagger$'s have been put next to their partner a's in the standard pattern, we have $\Sigma_i a_i^\dagger a_i = m$. Thus, ignoring the Kronecker deltas generated by moving $a^\dagger$'s past a's, we have $C_P = (-1)^P m^n$.

For $n \ll \ell$, most of the Kronecker delta's can indeed be neglected; the exception occurs when one of them has two identical indices, leading to $\Sigma_i \delta_{ii} = \ell$. To account for these dominant Kronecker deltas, we first note that, for a given permutation P, $\gamma_1(P)$ is the number of elements that are left in the same place by P, and these contribute no Kronecker deltas. Let $r(P)$ be the number of elements that are moved to the right of their original positions by P; only these $a^\dagger$'s need to pass through their partner a's on being returned to the standard order. For these, the factor of m must be replaced by $m-\ell$. Finally, the number of elements that are moved to the left of their original positions by P is $n - \gamma_1(P) - r(P)$, and these contribute no large Kronecker deltas on being returned to the standard order. Thus we have

$$C_P \cong (-1)^P \ell^{-n} m^{\gamma_1(P)} m^{n-\gamma_1(P)-r(P)} (m-\ell)^{r(P)} \tag{38}$$

for $n \ll \ell$.

Next we notice that the inverse permutation $\bar{P}$ has the counts of the left- and right-moving elements exchanged. Thus,

$$C_{\bar{P}} \cong (-1)^P \ell^{-n} m^{\gamma_1(P)} m^{r(P)} (m-\ell)^{n-\gamma_1(P)-r(P)}. \tag{39}$$

Then, using the fact that $(-1)^P$ and $\gamma_1(P)$ are each invariant under unitary transformations of P, we have

$$\begin{aligned} C_{\bar{R}PR} \cong (-1)^P \ell^{-n} m^{\gamma_1(P)} m^{r(\bar{R}PR)} \\ \times (m-\ell)^{n-\gamma_1(P)-r(\bar{R}PR)}. \end{aligned} \tag{40}$$

Setting Eqs. (38) and (40) into Eq. (36), we get

$$T_n \cong v_0^{2n}[m(m-\ell)]^n \ell^{-n} \times \sum_{P,R} \left(\frac{m}{m-\ell}\right)^{\gamma_1(P)-r(P)+r(\bar{R}PR)}. \quad (41)$$

The convention of Eq. (9) with $k=1$ yields $v_0^{2n}[m(m-\ell)]^n=(-1)^n$. Then we can write

$$T_n/(n!)^2 \cong \ell^{-n} M_n(f/(1-f)), \quad (42)$$

where $f=m/\ell$ is the filling fraction, and we have defined the function

$$M_n(x) \equiv \frac{(-1)^n}{(n!)^2} \sum_{P,R} (-x)^{\gamma_1(P)-r(P)+r(\bar{R}PR)}. \quad (43)$$

Remarkably, it is possible to evaluate $M_n(x)$ exactly for arbitrary n without performing the sum over the permutations P and R in Eq. (43) explicitly. This calculation is given in the Appendix. We find that $M_n(x)$ is everywhere positive and convex; it diverges as $x\to 0$ and $x\to+\infty$, and has a minimum between $x=0$ and $x=1$. While we are not able to evaluate the asymptotic form of $M_n(x)$ analytically, numerical evaluation reveals that, for large n,

$$M_n(x) \sim n\lambda^n, \quad (44)$$

where λ is a number that depends on x; if x is real and positive, so is λ. Using Eqs. (42) and (44) in Eq. (25), we find

$$F(y) \sim \frac{\lambda y/\ell}{(1-\lambda y/\ell)^2}, \quad (45)$$

which obviously has a singularity on the positive real axis at $y=+\ell/\lambda$. Hence the integral in Eq. (24) for $R(z_1^+,z_2^-)$ is not well defined. Therefore, we argue, the binary correlation method fails to give us the density-density correlation function.

IV. CONCLUSIONS

It is significant that, while we cannot compute $R(z_1^+,z_2^-)$ by the binary correlation method, we *can* compute $R(z_1^+,z_2^+)$. In this case (at least for $k=1$), there is no singularity in the integrand in Eq. (24), and so we can complete the integrals, take the $\ell\to\infty$ limit, and conclude that $R(z_1^+,z_2^+)=R(z_1^-,z_2^-)^*=0$. This is consistent with other approaches to the computation of this spectral correlation function. For example, in the periodic-orbit approach to chaotic systems, one finds (see, e.g., [13])

$$G(z^\pm)=\sum_{\text{p.o.}} w_p e^{\mp iS_p/\hbar} e^{-\varepsilon T_p}, \quad (46)$$

where S_p is the action of the orbit (including the contribution of the Maslov phase), T_p is its period, and w_p is the Gutzwiller weight factor. The key point is that, in the limit of small $\hbar$, $G(z_1^+)G(z_2^+)$ contains only large $\gg 2\pi$) phases, and so summing the orbits should give zero. On the other hand, $G(z_1^+)G(z_2^-)$ has negative phases from the first factor and positive from the second, so cancellations can occur; in the most straightforward approach (the so-called diagonal approximation), one simply keeps only those terms with an exact cancellation. This is adequate for computing the density-density correlation function at large $|E_1-E_2|$; for chaotic systems, it is nonzero, and agrees with the Wigner-Dyson prediction of random-matrix theory. Thus, for the k-body random-interaction model, the binary-correlation method correctly and unambiguously gives us $R(z_1^+,z_2^+)=0$, but, as we have argued, formal manipulations of divergent series are not adequate for extending this result to $R(z_1^+,z_2^-)$. We have seen explicitly how the binary-correlation approximation fails to give us a series for $R(z_1^+,z_2^-)$ that can be evaluated by Borel summation in the case $k=1$.

It is important to remember, though, that the binary correlation approximation breaks down precisely where we wish to apply it: when the number of contractions approaches the number of single-particle levels. Thus, to reach a firm conclusion one way or the other, we are forced to go beyond this approximation. At the present time, doing so appears to present a severe challenge to the available technologies.

One of these technologies, also investigated by BRW, is the supersymmetric sigma model [14]. BRW found that, at the tree and one-loop levels, the sigma model predicts Wigner-Dyson statistics for $k\ll m\ll\ell$, with corrections that go to zero as $1/\ell$. Unfortunately, this does not settle the matter, because this prediction also applies to $k=1$, where it is known to be false. One must therefore presume that higher-loop corrections are important.

Thus the only conclusion that seems completely safe at this juncture is a disappointing one: the nature of the spectral statistics of the k-body random-interaction model remains an unsolved problem.

ACKNOWLEDGMENTS

I am grateful to Hans Weidenmüller and (especially) Luis Benet for open and extensive discussions of their work. I also thank Felix Izrailev, Francois Leyvraz, and Thomas Seligman for discussions, and Steve Tomsovic for help with the combinatorics. I also thank Thomas Seligman for the kind hospitality of the Centro Internacional de Ciencias A.C., where this work was begun under the additional sponsorship of CONACyT, Mexico. This work was also supported in part by NSF Grant No. PHY99-70701.

APPENDIX

We wish to evaluate

$$(-1)^n(n!)^2 M_n(-x)=\sum_{P,R} x^{\gamma_1(P)-r(P)+r(\bar{R}PR)}, \quad (A1)$$

where P and R are permutations of n elements, $\gamma_1(P)$ is the number of elements left in their original positions by P, and $r(P)$ is the number of elements that are moved to the right of their original positions by P. For small values of n, we can

perform this sum over $(n!)^2$ terms explicitly. We are, however, interested in its behavior for large n, and so this strategy quickly becomes untenable.

Let us warm up by computing

$$R_n(x)=\sum_P x^{r(P)}. \tag{A2}$$

Let $N_{n,r}$ be the number of permutations of n elements in which exactly r of the elements move to the right. Then

$$R_n(x)=\sum_{r=0}^{n-1} N_{n,r}x^r. \tag{A3}$$

We can compute $N_{n,r}$ by the method of inclusion and exclusion [15]. Let $S_{n,r}$ be the number of permutations in which *at least* r elements move to the right. Then [15]

$$N_{n,r}=\sum_{k=0}^{n-r}(-1)^k\binom{r+k}{k}S_{n,r+k}. \tag{A4}$$

To obtain $S_{n,r}$, let us first count the number of permutations in which r *specific* elements move to the right, irrespective of what happens to the remaining elements. Actually, since we conventionally count from left to right, it is slightly more calculationally convenient to count the number of permutations that move r specific elements to the left, rather than to the right; clearly, by left-right symmetry, the choice of direction is irrelevant to the count. So, let A_{n,i_1} be the number of permutations in which the i_1^{th} element moves to the left. There are i_1-1 possible places for the i_1^{th} element to go, and $(n-1)!$ permutations of the remaining elements; thus

$$A_{n,i_1}=(i_1-1)(n-1)!. \tag{A5}$$

Similarly, let $A_{n,i_1,\ldots,i_r}$ with $i_1<\cdots<i_r$ be the number of permutations in which all the named elements move to the left; by a similar argument, we have

$$A_{n,i_1,\ldots,i_r}=(i_1-1)(i_2-2)\cdots(i_r-r)(n-r)!. \tag{A6}$$

Now $S_{n,r}$ is given by the sum over all possible values of the i's of $A_{n,i_1,\ldots,i_r}$:

$$\begin{aligned}S_{n,r}&=\sum_{i_r=r}^{n}\cdots\sum_{i_2=2}^{i_3-1}\sum_{i_1=1}^{i_2-1}A_{n,i_1,\ldots,i_r}\\&=(n-r)!\sum_{i_r=r}^{n}(i_r-r)\cdots\sum_{i_2=2}^{i_3-1}(i_2-2)\sum_{i_1=1}^{i_2-1}(i_1-1)\\&=(n-r)!\sum_{j_r=0}^{n-r}j_r\cdots\sum_{j_2=0}^{j_3}j_2\sum_{j_1=0}^{j_2}j_1,\end{aligned} \tag{A7}$$

where, in the last line, $j_a=i_a-a$. Equations (A3), (A4), and (A7) give us $R_n(x)$. The first few of these polynomials are

$$\begin{aligned}R_1(x)&=1,\\R_2(x)&=1+x,\\R_3(x)&=1+4x+x^2,\\R_4(x)&=1+11x+11x^2+x^3,\\R_5(x)&=1+26x+66x^2+26x^3+x^4.\end{aligned} \tag{A8}$$

Also, it turns out to be convenient later if we adopt the convention

$$R_0(x)=1/x. \tag{A9}$$

Now consider $\Sigma_R x^{r(\bar{R}PR)}$ for fixed P. Let P belong to cycle class $\mathcal{C}$, specified by the numbers γ_c of cycles of length c, with $\Sigma_{c=1}^{n}c\gamma_c=n$. The permutation $\bar{R}PR$ belongs to the same class (since class is preserved by unitary transformations), and furthermore each permutation in the class appears equally often as R is varied over the group. Thus

$$\sum_R x^{r(\bar{R}PR)}=\frac{n!}{N_{\mathcal{C}}}\sum_{P\in\mathcal{C}}x^{r(P)}, \tag{A10}$$

where

$$N_{\mathcal{C}}=\frac{n!}{\prod_{c=1}^{n}c^{\gamma_c}\gamma_c!} \tag{A11}$$

is the number of permutations in class $\mathcal{C}$ [15]. Let $\tilde{N}_{c,r}$ be the number of permutations of the elements in a particular cycle of length c in which exactly r of the elements in this cycle move to the right. A cycle can be specified by an ordering of its elements with the smallest first; thus a cycle of length c can be mapped to a permutation of the remaining $c-1$ elements. This mapping can be shown to imply $\tilde{N}_{c,r}=N_{c-1,r-1}$. Thus,

$$\sum_{r=1}^{c-1}\tilde{N}_{c,r}x^r=\sum_{r=1}^{c-1}N_{c-1,r-1}x^r=\sum_{s=0}^{c-2}N_{c-1,s}x^{s+1}=xR_{c-1}(x). \tag{A12}$$

The convention of Eq. (A9) correctly treats the case $c=1$. Also, the number of ways of assigning elements to the cycles is

$$N_A=\frac{n!}{\prod_{c=1}^{n}(c!)^{\gamma_c}\gamma_c!}. \tag{A13}$$

Thus

$$\sum_{P\in\mathcal{C}} x^{r(P)} = N_A \prod_{\text{cycles}} xR_{c-1}(x)$$

$$= N_A \prod_{c=1}^{n} [xR_{c-1}(x)]^{\gamma_c} = n! \prod_{c=1}^{n} \frac{1}{\gamma_c!}\left[\frac{xR_{c-1}(x)}{c!}\right]^{\gamma_c}. \tag{A14}$$

Therefore, by Eqs. (A10), (A11), and (A14),

$$\sum_{R} x^{r(\bar{R}PR)} = n! \prod_{c=1}^{n} \left[\frac{xR_{c-1}(x)}{(c-1)!}\right]^{\gamma_c}. \tag{A15}$$

Now we can write

$$(-1)^n (n!)^2 M_n(-x) = \sum_{P,R} x^{\gamma_1(P)-r(P)+r(\bar{R}PR)}$$

$$= \sum_{\mathcal{C}} \sum_{P\in\mathcal{C}} x^{\gamma_1(P)-r(P)} \sum_{R} x^{r(\bar{R}PR)}. \tag{A16}$$

The sum over R is given by Eq. (A15), and the sum over $P\in\mathcal{C}$ is given by Eq. (A14) with $xR_{c-1}(x)$ replaced by $x^{\delta_{c1}-1}R_{c-1}(1/x)$. The sum over classes is equivalent to a sum over all possible values of each γ_c with the constraint that $\Sigma_{c=1}^{n} c\,\gamma_c = n$. This is most easily treated via a generating function

$$\mathcal{M}(x,z) = \sum_{n=0}^{\infty} M_n(x) z^n. \tag{A17}$$

Then, from Eqs. (A14)–(A17) we have

$$\mathcal{M}(x,z) = \sum_{\gamma_c=0}^{\infty} \prod_{c=1}^{\infty} \frac{1}{\gamma_c!}\left[\frac{(-z)^c (-x)^{\delta_{c1}} R_{c-1}(-x) R_{c-1}(-1/x)}{c!(c-1)!}\right]^{\gamma_c}$$

$$= \exp\left[\sum_{c=1}^{\infty} \frac{(-z)^c (-x)^{\delta_{c1}} R_{c-1}(-x) R_{c-1}(-1/x)}{c!(c-1)!}\right]. \tag{A18}$$

The expansion of Eq. (A18) in powers of z yields $M_n(x)$ as the coefficient of z^n. The results for small n are

$$M_1(x) = x,$$

$$M_2(x) = (1+x^2)/2,$$

$$M_3(x) = (1/x - 2 + 7x + 2x^3)/(2!3!),$$

$$M_4(x) = (1/x^2 - 8/x + 48 - 32x + 49x^2 + 6x^4)/(3!4!). \tag{A19}$$

We have verified this procedure for computing $M_n(x)$ up through $n=7$ by comparison with the brute-force summation of the right-hand side of Eq. (A1).

It is tempting to conclude from Eq. (A19) that, in the limit of small x, we can take $M_n(x) = x^{-(n-2)}/(n!(n-1)!)$. Small x corresponds to small filling fraction $f = m/\ell$, which is what we want if we are interested in the limit $\ell\to\infty$ with m fixed. This limiting form for $M_n(x)$ would lead to convergence of the series in Eq. (25) for all y, with no singularities on the real axis, thus apparently validating the BRW result. However, including the first correction yields

$$M_n(x) = x^{-(n-2)}[1 - (2^n - 2n)x + \cdots]/(n!(n-1)!), \tag{A20}$$

and so we see that we must have $x \ll 2^{-n}$ before the first term alone is an adequate approximation. Instead, for the purpose of determining the convergence of the series in Eq. (25), we should take the limit of large n with x held fixed. We have not found a way to do this analytically, and so have resorted to numerical methods. For a given value of x, we can evaluate $M_n(x)$ up to around $n=500$ from Eq. (A18) in a reasonable amount of computation time. We find, for $x=0.001$, 0.01, 0.1, 0.5, and 1.0, that the behavior of $M_n(x)$ for $n > 10$ is very well fit by Eq. (44), with $\lambda = 17.4$, 3.28, 0.797, 0.435, and 0.405, respectively.

[1] L. Benet, T. Rupp, and H. A. Weidenmüller, Phys. Rev. Lett. **87**, 010601 (2001).

[2] L. Benet, T. Rupp, and H. A. Weidenmüller, Ann. Phys. (N.Y.) **292**, 67 (2001).

[3] J. B. French and S. S. M. Wong, Phys. Lett. B **35**, 5 (1971).

[4] O. Bohigas and J. Flores, Phys. Lett. B **34**, 261 (1971).

[5] K. K. Mon and J. B. French, Ann. Phys. (N.Y.) **95**, 90 (1975).

[6] T. Asaga, L. Benet, T. Rupp, H. A. Weidenmüller, Europhys. Lett. **56**, 340 (2001); Ann. Phys. (N.Y.) **298**, 229 (2002).

[7] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).

[8] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. **299**, 189 (1998).

[9] L. Benet, T. Rupp, and H.A. Weidenmüller (private communi-

cation).
[10] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[11] G. Casati, F. Valz-Gris, and I. Guarnieri, Lett. Nuovo Cimento Soc. Ital. Fis. **28**, 279 (1980).
[12] J. J. M. Verbaarschot and M. R. Zirnbauer, Ann. Phys. (N.Y.) **158**, 78 (1984).
[13] E. Bogomolny, in *New Directions in Quantum Chaos: Proceedings of the International School of Physics "Enrico Fermi," Course CXLIII*, edited by G. Casati, I. Guarneri, and U. Smilansky (IOS Press, Amsterdam, 2000).
[14] K. B. Efetov, Adv. Phys. **32**, 53 (1983).
[15] J. Riordan, *An Introduction to Combinatorial Analysis* (Wiley, New York, 1958).